SOME EFFECTS RESULTING FROM THE ANISOTROPY OF THE ELECTRICAL CONDUCTIVITY IN A GAS-DISCHARGE PLASMA

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We consider certain effects of the anisotropy of electrical conductivity in a glow discharge plasma to which a uniform external magnetic field is applied.

The effect of the inclination of the external magnetic field upon the current distribution at an electrode located in a plane, semi-infinite channel is considered in the first section, and a general solution is derived. Calculations are performed for some special cases. It was found that the inclination of the magnetic field relative to the electrode surface causes a sharp inhomogeneity in the current distribution.

The second section deals with the current in a plasma characterized by a given nonuniform charge carrier density; the charge carriers are assumed to be in a plane channel with nonconductive walls. It is shown that the plasma inhomogeneity strongly influences the anisotropy parameter dependence of the following ratio: Hall current/current in the direction of the electric field.

When both the magnetic Reynolds number $R_m$ and the interaction parameter S are small, the electric current density and the electric field strength are given by the equations

$$\mathbf{j}+\omega\tau(\mathbf{j}\times\mathbf{l})=\frac{e\tau}{m_e}(en\mathbf{E}+kT_e\nabla n);\quad \nabla\mathbf{j}=0;\quad \nabla\times\mathbf{E}=0\,. \quad (0.1)$$

The unit vector of the external magnetic field is denoted by l, and the concentration of the electrons by n, which is assumed to be a known function of the coordinates.

We assume that the magnetic force lines are parallel to the xy-plane and that all quantities are independent of the z coordinate. It follows from the last equation of (0.1) that $E_z$ = const.

After eliminating the electric field strength E from the first equation, we obtain

$$\nabla\times\mathbf{j}+\omega\tau(l\nabla)\,\mathbf{j}=\nabla\ln n\times\mathbf{j}+\omega\tau\nabla\ln n\times(\mathbf{j}\times\mathbf{l})\,. \quad (0.2)$$

By expressing the components $j_x$ and $j_y$ of the current density in terms of the stream function $\psi$, we obtain from Eqs. (0.1) and (0.2)

$$(1+\omega^2\tau^2\cos^2\theta)\frac{\partial^2\psi}{\partial x^2}+2\omega^2\tau^2\cos\theta\sin\theta\frac{\partial^2\psi}{\partial x\,\partial y}+$$
$$+(1+\omega^2\tau^2\sin^2\theta)\frac{\partial^2\psi}{\partial y^2}-$$
$$-\left[\frac{\partial\ln n}{\partial x}+\omega^2\tau^2\left(\cos\theta\frac{\partial\ln n}{\partial x}+\sin\theta\frac{\partial\ln n}{\partial y}\right)\cos\theta\right]\frac{\partial\psi}{\partial x}-$$
$$-\left[\frac{\partial\ln n}{\partial y}+\omega^2\tau^2\left(\cos\theta\frac{\partial\ln n}{\partial x}+\sin\theta\frac{\partial\ln n}{\partial y}\right)\sin\theta\right]\frac{\partial\psi}{\partial y}=0;$$
$$\cos\theta=l_x,\quad \sin\theta=l_y,\quad j_x=\partial\psi/\partial y,\quad j_y=-\partial\psi/\partial x\,. \quad (0.3)$$

In the following, Eq. (0.3) is used in two specific problems.

**1.** The effect of the inclination of the magnetic field vector relative to the axis of an infinitely long channel with nonconductive walls upon the diffusion of a glow discharge plasma was discussed in [1,2]. The authors of [1,2] ignored boundary effects resulting from the electrodes.

We will discuss how the inclination of the magnetic field affects the current distribution at an electrode. We assume that a plane semi-infinite channel is formed by two nonconductive walls with the coordinates y = 0 and y = b and by an electrode (at x = 0) with ideal conduction; the other electode is assumed to be located at infinity. When we assume that the electron concentration is constant, we obtain the following equation for $\psi$ from Eq. (0.3):

$$(1+\omega^2\tau^2\cos^2\theta)\frac{\partial^2\psi}{\partial x^2}+2\omega^2\tau^2\cos\theta\sin\theta\frac{\partial^2\psi}{\partial x\,\partial y}+$$
$$+(1+\omega^2\tau^2\sin^2\theta)\frac{\partial^2\psi}{\partial y^2}=0\,. \quad (1.1)$$

The fact that the tangential component of the electric field strength vanishes at the electrode, and that the normal component of the current density vanishes at the nonconductive walls leads to the following boundary conditions for $\psi$:

$$\left[(1+\omega^2\tau^2\cos^2\theta)\frac{\partial\psi}{\partial x}+\omega^2\tau^2\cos\theta\sin\theta\frac{\partial\psi}{\partial y}\right]_{x=0}=0\,, \quad (1.2)$$

$$\psi|_{y=0}=0;\quad \psi|_{y=b}=I;\quad I=\int_0^b j_x\,dy\,. \quad (1.3)$$

When expressed by the dimensionless variables

$$\xi=\frac{x}{\beta b};\quad \eta=\frac{y}{b};\ u=\frac{\psi}{I};\quad \beta=\frac{\sqrt{1+\omega^2\tau^2}}{1+\omega^2\tau^2\sin^2\theta}\,.$$

Eq. (1.1) and the boundary conditions (1.2) and (1.3) assume the form

$$(1+\gamma^2)\frac{\partial^2u}{\partial\xi^2}+2\gamma\frac{\partial^2u}{\partial\xi\,\partial\eta}+\frac{\partial^2u}{\partial\eta^2}=0\left(\gamma=\frac{\omega^2\tau^2\cos\theta\sin\theta}{\sqrt{1+\omega^2\tau^2}}\right); \quad (1.4)$$

$$\left[(1+\gamma^2)\frac{\partial u}{\partial\xi}+\gamma\frac{\partial u}{\partial\eta}\right]_{\xi=0}=0;\quad u|_{\eta=0}=0,\quad u|_{\eta=1}=1\,. \quad (1.5)$$

We introduce the new variables

$$\xi_1=\eta-\frac{\gamma\xi}{1+\gamma^2};\qquad \eta_1=\frac{\xi}{1+\gamma^2}\,, \quad (1.6)$$

which transform Eq. (1.4) into the Laplace equation and Eq. (1.5) into a condition of the form

$$u|_{\xi_1+\gamma\eta_1=0}=0;\quad u|_{\xi_1+\gamma\eta_1=1}=1;\quad \left.\frac{\partial u}{\partial\eta_1}\right|_{\eta_1=0}=0\,. \quad (1.7)$$

The relations

$$\frac{dz}{d\zeta}=C\zeta^{-\beta/\pi}(\zeta-1)^{-(\pi-\beta)/\pi};\quad \mathrm{tg}\,\beta=\frac{1}{\gamma}\,, \quad (1.8)$$

are used for a conformal mapping of the region (considered in the plane $Z=\xi_1+i\eta_1$) onto the half-plane $\mathrm{Im}\zeta>0$. It is convenient, considering the following discussion, to project the half-plane onto the strip $0<\mathrm{R}\lambda<1$ by introducing the relation

$$\lambda=\frac{2}{\pi}\arcsin\sqrt{\zeta}\,. \quad (1.9)$$

Since, in the $\lambda$-plane of the complex potential, we have, for reasons of symmetry,

$$w=u+iv=\lambda\,, \quad (1.10)$$

we finally obtain the solution in the form

$$z=\left[\int_0^w\left(\mathrm{tg}\frac{\pi w}{2}\right)^{1-2\mu}dw\right]\left[\int_0^1\left(\mathrm{tg}\frac{\pi w}{2}\right)^{1-2\mu}dw\right]^{-1}\quad\left(\mu=\frac{\beta}{\pi}\right). \quad (1.11)$$

For $\mu=1/2$, when $\gamma=0$ (this occurs either for $\omega\tau=0$, for $\theta=0$, or for $\theta=\pi/2$), Eq. (1.11) leads to a linear relation, $u=\eta$. Thus, when the magnetic field is parallel or perpendicular to the surface of the electrode, the anisotropy of the conductivity does not influence the current density (which remains uniform for all $\omega\tau$).

The effect resulting from an inclination of the magnetic force lines relative to the electrode surface can easily be derived for the particular case $\mu=1/4$ ($\gamma=1$). The integrals in Eq. (1.11) may then be calculated, and the solution assumes the form

$$z = \frac{1}{\pi}\left[\operatorname{arc\,tg}\frac{\sqrt{2\operatorname{tg}{}^1/_2\pi w}}{1-\operatorname{tg}{}^1/_2\pi w} - \frac{1}{2}\ln\frac{1+\sqrt{2\operatorname{tg}{}^1/_2\pi w}+\operatorname{tg}{}^1/_2\pi w}{1-\sqrt{2\operatorname{tg}{}^1/_2\pi w}+\operatorname{tg}{}^1/_2\pi w}\right]. \tag{1.12}$$

Accordingly, we obtain on the electrode surface

$$\eta = \frac{1}{\pi}\left[\operatorname{arc\,tg}\frac{\sqrt{2\operatorname{tg}{}^1/_2\pi u}}{1-\operatorname{tg}{}^1/_2\pi u} - \frac{1}{2}\ln\frac{1+\sqrt{2\operatorname{tg}{}^1/_2\pi u}+\operatorname{tg}{}^1/_2\pi u}{1-\sqrt{2\operatorname{tg}{}^1/_2\pi u}+\operatorname{tg}{}^1/_2\pi u}\right], \tag{1.13}$$

which leads to the following formula for the dimensionless normal component of the current density at the electrode:

$$j = \frac{j_x b}{I} = \frac{\partial u}{\partial \eta}\bigg|_{\xi=0} = \sqrt{2\operatorname{ctg}{}^1/_2\pi u}\,. \tag{1.14}$$

The case $\gamma \ll 1$ is easy to discuss. When we represent the solution in the form of a power series

$$u = u_0 + \gamma u_1 + \gamma^2 u_2 + \ldots,$$

we obtain the equations

$$\frac{\partial^2 u_0}{\partial \xi^2} + \frac{\partial^2 U_0}{\partial \eta^2} = 0, \qquad \frac{\partial^2 u_1}{\partial \xi^2} + \frac{\partial^2 u_1}{\partial \eta^2} = -2\frac{\partial^2 u_0}{\partial \xi \partial \eta} \tag{1.15}$$

for $u_0$ and $u_1$ with the boundary conditions

$$\begin{aligned} &u_0 = 0, && u_1 = 0 && \text{for } \eta = 0 \\ &u_0 = 1, && u_1 = 0 && \text{for } \eta = 1 \\ &\frac{\partial u_0}{\partial \xi} = 0, && \frac{\partial u_1}{\partial \xi} = -\frac{\partial u_0}{\partial \eta} && \text{for } \xi = 0. \end{aligned} \tag{1.16}$$

The series

$$u = \eta + 2\gamma\sum_{k=1}^{\infty}\frac{1-(-1)^k}{(k\pi)^2}\sin k\pi\eta\, e^{-k\pi\xi}, \tag{1.17}$$

is the solution to Eq. (1.15) for the conditions stated in Eq. (1.16).

This results in the formula

$$j\,|_{\xi=0} = 1 - \frac{2\gamma}{\pi}\ln\operatorname{tg}\frac{1}{2}\pi\eta \tag{1.18}$$

for the current density at the electrode.

The figure with the results calculated for $\gamma = 0$, 0.1, and 1.0 indicates that an inclination of the magnetic field causes a strong inhomogeneity in the current distribution at the electrode.

**2.** The results of Hall current measurements in a homopolar conductor were published in [3]. It was noted that the experimental current values are many times smaller than the theoretical values. The authors explained the discrepancy by inhomogeneities in the discharge which prevent the free flow of the Hall current.

This conclusion can be easily verified by the following simple example. Let us assume that in the rectangular region $-a \leq x \leq a$, $0 \leq y \leq b$, the concentration of the charged particles changes according to the law $n = n_0 e^{-\alpha|x|}$ and that the magnetic field is parallel to the y-axis. In this case, Eq. (0.3) assumes the form

$$\frac{\partial^2\psi}{\partial x^2} + (1+\omega^2\tau^2)\frac{\partial^2\psi}{\partial y^2} \mp \alpha\frac{\partial\psi}{\partial x} = 0, \tag{2.1}$$

where the minus sign must be taken for x > 0 and the plus sign for x < 0. When the condition

$$j_y = -\frac{\partial\psi}{\partial x} = 0$$

is satisfied at the boundaries, the solution to Eq. (2.1) is

$$\psi = \psi_b y/b, \tag{2.2}$$

where $\psi = 0$ was assumed at y = 0.

Equation (2.2) results in

$$j_x = \frac{\psi_b}{b}; \quad j_y = 0; \quad E_y = 0; \quad E_x = E_x(x).$$

By integrating (over x and y) the projection of the first equation of (0.1) onto the Z-axis, we obtain

$$I_z + \omega\tau I_x = \frac{e^2\tau E_0}{m_e}2abn_0\left(\frac{1-e^{-\alpha a}}{\alpha a}\right), \tag{2.3}$$

where

$$I_z = \int_{-a}^{a}\int_0^b j_z dxdy; \quad I_x = \int_{-a}^{a}\int_0^b j_x dxdy; \quad E_0 = E_z.$$

On the other hand, integrating the expression

$$\frac{j_x}{n} = \frac{e\tau/m_e}{1+\omega^2\tau^2}\left[eE_x + \frac{kT_e}{n}\frac{dn}{dx} + \omega\tau eE_0\right]$$

over x under the conditions

$$n\,|_{x=-a} = n\,|_{x=a}; \quad \varphi\,|_{x=-a} = \varphi\,|_{x=a} \quad \left(\frac{d\varphi}{dx} = -E_x\right)$$

we obtain

$$j_x = \frac{\omega\tau}{1+\omega^2\tau^2}\frac{e^2\tau E_0 n_0}{m_e}\left(\frac{\alpha a}{e^{\alpha a}-1}\right). \tag{2.4}$$

By calculating $I_x = 2abj_x$ and inserting into Eq. (2.3), we obtain

$$I_z = \frac{e^2\tau E_0 2abn_0}{m_e}\left[\frac{1-e^{-\alpha a}}{\alpha a} - \frac{\omega^2\tau^2}{1+\omega^2\tau^2}\frac{\alpha a}{e^{\alpha a}-1}\right]. \tag{2.5}$$

Thus, we obtain for the ratio of Hall current $I_x$ to basic current $I_z$,

$$\frac{I_x}{I_z} = \frac{\omega\tau(\alpha a)^2}{2(\operatorname{ch}\alpha a - 1)(1+\omega^2\tau^2) - (\omega\tau\alpha a)^2}. \tag{2.6}$$

It follows from Eq. (2.6) that the $\omega\tau$ dependence of $I_x/I_z$ varies with the inhomogeneity parameter $\alpha a$. For $\alpha a \to 0$ we obtain an expression which is well known for a uniform plasma [3]:

$$I_x / I_z = \omega\tau. \tag{2.7}$$

On the other hand, we obtain for $\alpha a \gg 1$

$$\frac{I_x}{I_z} = \frac{\omega\tau}{1+\omega^2\tau^2}\frac{(\alpha a)^2}{2\operatorname{ch}\alpha a}. \tag{2.8}$$

If $\omega\tau > 1$, Eqs. (2.8) and (2.7) both yield $I_x/I_z$ as a linear function of $\omega\tau$, but with proportionality factor

$$\frac{(\alpha a)^2}{2\operatorname{ch}\alpha a} \ll 1.$$

If, on the other hand, $\omega\tau \gg 1$, $I_x/I_z$ is inversely proportional to $\omega\tau$.

We note, in conclusion, that the effect of a one-dimensional conductivity inhomogeneity on the power of a magnetohydrodynamic generator and on the nonisothermal ionization was previously discussed in [4].

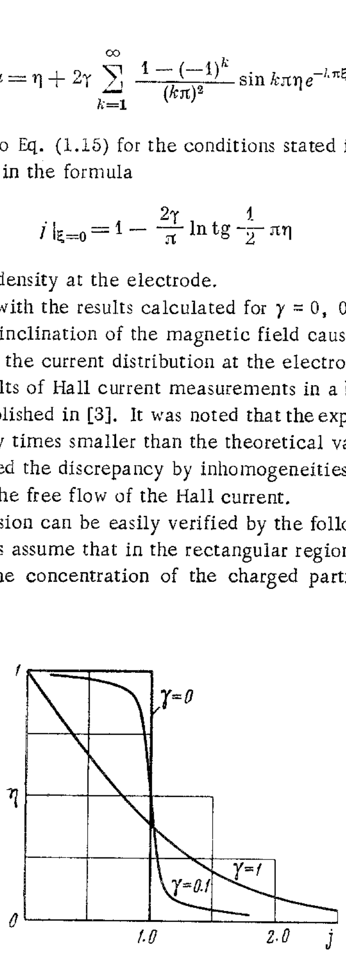


Fig. 1.

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